EXCITATIONS AND DEVELOPMENT OF INSTABILITIES

IN THREE-DIMENSIONAL STATIONARY BOUNDARY LAYERS

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UDC 532.526.013

1. Statement of the Problem. The amplitude method of determining the critical Reynolds numbers of the transition of a laminar boundary layer into a turbulent one in the case of sufficiently low amplitudes of initial perturbations reduces, as is well known [1, 2], to the study of perturbation development in the linearized theory of hydrodynamic stability until the region on the aircraft surface, where the threshold value is reached for the perturbation amplitude $\varepsilon^* = (\alpha \mathrm{Re}_\delta)^{-2/3}$ (α is the wave number for perturbations in gauges of the boundary layer width, and Re_δ is the Reynolds number, calculated from the width and local parameters of the boundary layer).

In a number of cases it can be assumed that $\mathrm{Re}^* = u_\infty L^*/\nu$ (L^* is the coordinate along the flow direction, corresponding to the location where the perturbation amplitude reaches the value ε^*) is approximately equal to the corresponding critical Reynolds number of transition to turbulence. Thus, the approximate calculation of critical Reynolds numbers reduces then to the study of excitations and development of instability in the linearized case.

Quite important is here the problem of excitation of instabilities always occurring in the flow of background perturbation (initial turbulence in the leading flow, surface roughness of an aircraft, its vibration, various kinds of its chemical inhomogeneities, etc.). The problem of instability excitation has attracted increasing attention in recent years, as is well known (see, for example, [2-8]).

Certain aspects of this problem are considered in the present study.

The assumption of a narrow layer in which instabilities develop is basic for the following treatment. We choose a curvilinear coordinate system xyz (Fig. 1), such that the boundary layer is located near the surface S, corresponding to $y = 0$. If the stream flow problem is considered, the surface S is the stream surface.

We denote the width of the boundary layer by δ, and let the characteristic sizes along the x and z axes be identical and equal to L. The assumption about the narrowness of the layer is written in the form

$$\varkappa = \delta/L \ll 1.$$

We define the vector Q as the matrix column Q_j ($1 \leqslant j \leqslant k$), consisting of the velocity projections onto the directions of the x, y, z axes, as well as of temperature, pressure, density, concentration components of the medium, the components of the electric and magnetic fields, and their first derivatives with respect to x, y, and z.

We introduce a perturbation for the quantity Q:

$$\mathbf{Q} = \mathbf{Q}_0 + \varepsilon \mathbf{A}.$$

The quantity Q_0 describes the stationary flow field of the medium in the boundary layer adjacent to the surface S, and is the exact solution of the original hydrodynamic equations.

The quantity ε will be assumed to be rather small ($\varepsilon \ll 1$), so that to determine the quantity **A** one can use the linearized equations of motion of the medium.

The equation for the perturbation **A** can be represented in the operator form

$$\widehat{H}_t \frac{\partial \mathbf{A}}{\partial t} + \widehat{H}_x \frac{\partial \mathbf{A}}{\partial x} + \widehat{H}_z \frac{\partial \mathbf{A}}{\partial z} = \widehat{H}_0' \mathbf{A} + \widehat{H}_1' \mathbf{A}. \quad (1.1)$$

The operators $\hat{H}$ are represented by a matrix with elements H_{nm} ($1 \leqslant m,\ n \leqslant k$), and $\hat{H}_t$ — by a constant operator 0 or 1. The operators $\hat{H}_x$, $\hat{H}_z$ are multiplication operators; they contain

Moscow. Translated from Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 4, pp. 100-110, July-August, 1983. Original article submitted July 6, 1982.

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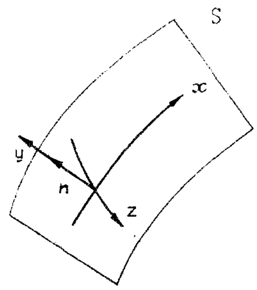

Fig. 1

the original flow characteristics, except the velocity component along the y axis, V. The operator $\hat{H}_0'$ contains the original flow characteristics except V, their derivatives with respect to y, and the operation $\partial/\partial y$. The multiplication operator $\hat{H}_1'$ contains derivatives of the original flow characteristics with respect to x, y, z, except the velocity V and its derivatives.

Equations (1.1) are a homogeneous system of k linear differential equations in first order partial derivatives of the k functions A_j (t, x, y, z) with varying coefficients, being known functions of x, y, z.

We dwell on the boundary conditions to be used in solving Eqs. (1.1). We will consider one of the following three types of boundary conditions:

1) The principal part of the quantity A_j vanishes for $y=0$, while quantities A_j also tend to zero for $y\to\infty$. We assume that this problem coincides with the frequently investigated problem of searching eigenfunctions and eigenvalues of the system (1.1).

2) The principal part of the quantity A_j vanishes for $y=0$, while the quantities A_j remain bounded for $y\to\infty$. These solutions, along with the solutions with conditions 1, contain incoming waves, related to the existence of perturbations in the leading flow.

3) The quantities A_j tend to zero for $y\to\infty$, while A_j remain bounded at $y=0$. This problem, along with the solutions with conditions 1), contain waves arriving at the boundary layer, generated by the presence of roughness, suction, vibrations, or any other perturbations proceeding from the surface flow.

2. General Method of Investigation (eikonal method). Let the unknown vector **A** be represented in the form of a Fourier integral

$$\mathbf{A} = \int_{(\omega)} \mathbf{A}_\omega (x, y, z, \omega)\, e^{i\omega t} dt. \tag{2.1}$$

We seek a solution for $\mathbf{A}_\omega$ in the form

$$\mathbf{A}_\omega = \mathbf{\Phi}(x, y, z, \omega) e^{F(x,z)}, \tag{2.2}$$

where the eikonal F(x, z) is

$$F(x, z) = -i \int_{\mathscr{L}} \alpha(x, z)\, dx + \gamma(x, z)\, dz. \tag{2.3}$$

The integration in (2.3) is carried out along some curve $\mathscr{L}$ in the x-z plane, while the upper limit of the xz integration is shifted by this curve, so that the eikonal F is a function of this upper limit. The integrand of expression (2.3) may generally not be a total differential.* Substituting expressions (2.1), (2.2) into system (1.1), we obtain a system of equations for determining the vector Φ:

$$-i\hat{H}_x \alpha \mathbf{\Phi} - i\hat{H}_z \gamma \mathbf{\Phi} = \hat{H}_0 \mathbf{\Phi} + \hat{H}_1 \mathbf{\Phi}, \tag{2.4}$$

where $\hat{H}_0 = \hat{H}_0' - i\omega \hat{H}_t$; $\hat{H}_1 = \hat{H}_1' - \hat{H}_x \frac{\partial}{\partial x} - \hat{H}_z \frac{\partial}{\partial z}$.

*The quantity $\alpha dx + \gamma dz$ is a total differential, for example, in the cases of two-dimensional flow in a boundary layer (when $\gamma=$const, and $\alpha=\alpha(x)$) or flow in a boundary layer of a slipping airfoil.

We make the following assumptions concerning orders of magnitude of the functions and the derivatives, appearing in (2.2), (2.3), corresponding to the basic assumption of a narrow layer. The characteristic length scales of the function Φ are the following: Along the x and z axes the scale is L, and along the y axis it is δ, so that we have the following derivative ratio:

$$\frac{\partial \Phi_j}{\partial x} \bigg/ \frac{\partial \Phi_j}{\partial y} \sim \varkappa, \ \frac{\partial \Phi_j}{\partial x} \sim \frac{\partial \Phi_j}{\partial z}. \tag{2.5}$$

Let the quantities α and γ be of the order of $1/\lambda$, and their derivatives

$$\partial\alpha/\partial x \sim \partial\alpha/\partial z \sim \partial\gamma/\partial x \sim \partial\gamma/\partial z \sim 1/L\lambda, \tag{2.6}$$

where λ is of the order of the wave length of the perturbation considered.

Assumptions (2.5), (2.6) correspond to slow changes of the functions Φ_j, α, γ along the x and z axes and a fast change of the functions Φ_j along the y axis.

In what follows the basic smallness parameter is the quantity

$$h = \lambda/L \ll 1 \ (\lambda = c_\lambda \delta)$$

(c_λ is a quantity of the order of magnitude between unity and ten).

If now the system of equations (2.4) is reduced to dimensionless form, for the portion of terms referring to the operator $\hat{H}_1$ there appears a small parameter h, i.e., in dimensionless form the equations acquire the form

$$\hat{L}\Phi = h\hat{H}_1\Phi \ (\hat{L} \equiv -i\hat{H}_x\alpha - i\hat{H}_z\gamma - \hat{H}_0). \tag{2.7}$$

The solution of system (2.7) is represented in the form of a power series in the small parameter

$$\Phi = \Phi^{(0)} + h\Phi^{(1)} + h^2\Phi^{(2)} + \ldots \tag{2.8}$$

Substituting the series (2.8) into (2.7), we obtain after equating terms of the order of smallness the following recurrent system of equations

$$\hat{L}\Phi^{(0)} = 0, \ \hat{L}\Phi^{(1)} = \hat{H}_1\Phi^{(0)}, \ldots \tag{2.9}$$

The system of equations (2.9) is a system of first-order ordinary differential equations in y, with the arguments x, z appearing in them as parameters.

The expansion (2.8) is an internal expansion for the problem under investigation.

The problem of solving system (2.9) for the zeroth approximation with the corresponding boundary conditions will be called the locally homogeneous problem. We assume that the locally homogeneous problem has been solved. This implies that we found a system of eigenvectors $\Phi(x, y, z)$, for which the complex quantity $\alpha(x, z)$ has both a continuous and a discrete eigenvalue spectrum. The quantity α classifies the possible types of perturbation. Let the dispersion relation be

$$\alpha = \alpha(x, z, a_i, \omega, \gamma), \tag{2.10}$$

where a_i is the set of physical parameters of the problem (the Reynolds and Mach numbers, etc.). Similarly to the quantity γ, the quantity ω is assumed given, therefore the equation for the zeroth approximation is directly written in the form

$$-i\hat{H}_x\alpha\Phi_\alpha = \hat{H}_{00}\Phi_\alpha \ (\hat{H}_{00} = \hat{H}_0 + i\hat{H}_z\gamma). \tag{2.11}$$

We define the product of two vectors

$$(\mathbf{A}, \mathbf{B}) = \lim_{\varepsilon\to 0} \int_0^\infty f(\varepsilon, y) \sum_{(n)} A_n B_n dy. \tag{2.12}$$

The function $f(\varepsilon, y)$ is chosen to be unity for problem 1, $\exp(-\varepsilon y)$ for problem 2, and $\exp(-\varepsilon/y)$ for problem 3.

We define the operator $\hat{H}^*$ as associate to the operator $\hat{H}$ if

$$(\hat{H}\mathbf{A}, \mathbf{B}) = (\mathbf{A}, \hat{H}^*\mathbf{B}). \tag{2.13}$$

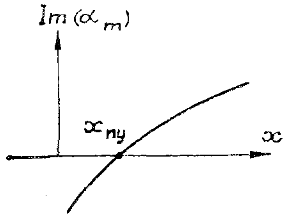

Fig. 2

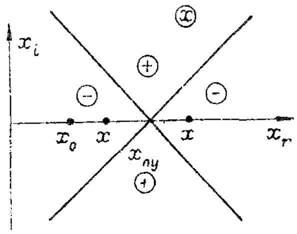

Fig. 3

We define the system of vectors Ψ_β is the solution of the associated problem, i.e., satisfying the equations

$$-i\hat{H}_x^*\beta\Psi_\beta = \hat{H}_{00}^*\Psi_\beta \tag{2.14}$$

and the corresponding boundary conditions, obtained by the standard procedure simultaneously with the shape of the associated operator $\hat{H}^*$ according to definition (2.13): The expression on the left-hand side of (2.13) is integrated in parts, so that only terms with integrals remain.

Multiplying (2.11) by Ψ_β, and taking into account (2.14), we have

$$(\alpha - \beta)(\hat{H}_x\Phi_\alpha, \Psi_\beta) = 0. \tag{2.15}$$

From expression (2.15) follows the orthogonality condition of the vectors Φ_α and Ψ_β for $\alpha \neq \beta$:

$$(\hat{H}_x\Phi_\alpha, \Psi_\beta) = 0. \tag{2.16}$$

We assume that the vector Φ_α is normalized, i.e.,

$$(\hat{H}_x\Phi_\alpha, \Psi_\beta) = \delta_{\alpha\beta} \tag{2.17}$$

($\delta_{\alpha\beta}$ is the Kronecker symbol) for the discrete spectrum, and

$$(\hat{H}_x\Phi_\alpha, \Psi_\beta) = \delta(\alpha - \beta) \tag{2.18}$$

($\delta(x)$ is the δ-function) for the continuous spectrum.

We turn now to costructing the external expansion for the problem under investigation. We represent the solution for $\mathbf{A}_\omega$ in the form

$$\mathbf{A}_\omega = \sum_{(n)} c_n(x, z)\,\Phi_n e^{F_n}, \tag{2.19}$$

where the quantity n counts all eigenvectors of the problem investigated. Substituting (2.19) into (2.1), (1.1), we obtain the expression

$$\sum_{(n)} \frac{\partial c_n}{\partial x}\hat{H}_x\Phi_n e^{F_n} + \sum_{(n)} \frac{\partial c_n}{\partial z}\hat{H}_z\Phi_n e^{F_n} = \sum_{(n)} c_n(\hat{H}_1\Phi_n)\,e^{F_n}. \tag{2.20}$$

Let now (for simplicity) the spectrum of eigenvectors considered be discrete. Multiplying (2.20) by the quantity Ψ_m (one of the corresponding vectors of the associated problem), and taking into account the orthogonality condition (2.17), we obtain the following system of equations in partial derivatives for determining the quantities $c_n(x, z)$ in the main approximation:

$$\frac{\partial c_m}{\partial x} + \sum_{(n)} \frac{\partial c_n}{\partial z} V_{mn} e^{F_n - F_m} = \sum_{(n)} c_n W_{mn} e^{F_n - F_m}, \tag{2.21}$$

where $V_{mn} = (\hat{H}_z\Phi_n, \Psi_m)$; $W_{mn} = (\hat{H}_1\Phi_n, \Psi_m)$. The number of equations in system (2.21) coincides with the number of functions $c_n(x, z)$. Relationship (2.21) can be treated as the solvability condition of system (2.9) for the following approximation, if a solution is sought in the form (2.19). The matrices V_{mn} and W_{mn} are logically called transition matrices. The diagonal terms (m = n) of the matrix W_{mn} describe the perturbation enhancement in the boundary layer due to the inhomogeneity in x and z. The nondiagonal terms (m ≠ n) describe the excitation process of mode m by mode n, i.e., they are generation sources of mode m. The existence of a nonvanishing matrix W_{mn} is exclusively related with the inhomogeneity of the original boundary layer along the x and z axes. The nondiagonal terms of the matrix V_{mn} also describe the generation process of mode m due to inhomogeneity of mode n in z.

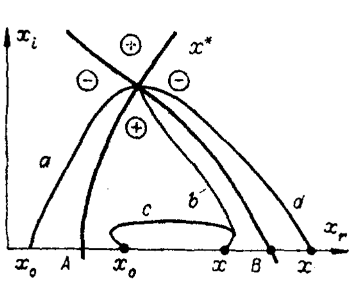

Fig. 4

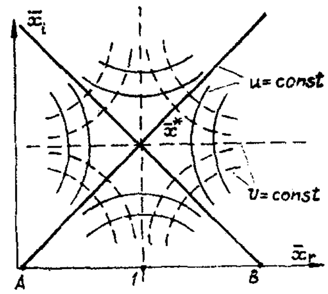

Fig. 5

When there exists a single-mode regime, i.e., $c_1 = c$, $c_2 = c_3 = \ldots = 0$, system (2.21) reduces to one first-order partial differential equation:

$$\partial c/\partial x + V_{11}\partial c/\partial z = W_{11}c. \tag{2.22}$$

Equation (2.22) coincides with the equation obtained in [9]. It must be noted that a single-mode regime does not always exist, and is usually an exception. Similarly determined are also the following approximations in h for the solution sought.

The eikonal method presented is a typical perturbation method. Formally it resembles the theory of quantum transitions for time-dependent perturbations (see, for example, [10]). An important difference is that the eikonal has a real part, and for problems of instability development the eikonal as an analytic function has, at least, one saddle point, so that, as seen below, it determines the basic character of the solution investigated.

If there exist simultaneously discrete and continuous spectra, instead of (2.19) one must write

$$\mathbf{A}_\omega = \sum_{(n)} c_n(x, z)\,\Phi_n e^{F_n} + \int_{(\alpha)} c_\alpha(x, z)\,\Phi_\alpha e^{F_\alpha} d\alpha. \tag{2.23}$$

The generalization of Eq. (2.21) to the case (2.23) can be obtained without difficulty.

3. Study of the Planar Case. Consider first the case in which the main flow in the boundary layer is two-dimensional and is independent of the coordinate z. The system of equations (2.21) is rewritten in the form

$$\frac{\partial c_m}{\partial x} = \sum_{(n)} c_n W_{mn} e^{F_n - F_m} \left(F_k = -i\int_{x_1} \alpha_k(x)\,dx\right). \tag{3.1}$$

In the one-dimensional case $c_1 = c$, $c_2 = c_3 = \ldots = 0$ we have an equation for the change in the wave amplitude

$$dc/dx = cW(x),$$

whence

$$\mathbf{A}_\omega = c_0 e^{\int_{x_0}^{x} W(x)dx}\,\Phi(x, y)\,e^{F(x)}.$$

A well-known solution was obtained (see, for example, [11]) for wave propagation, in particular the Tollmien—Schlichting wave, with account of the non-parallel nature of the main flow, when the wave amplitude is known at the point $x = x_0$.

Consider now the two-dimensional regime: For two fixed m and n $c_n \neq 0$, $c_m \neq 0$, and the remaining modes are absent. The system (3.1) transforms to a system of two equations

$$\begin{aligned} \partial c_m/\partial x &= c_m W_{mm} + c_n W_{mn} e^{\Delta}, \\ \partial c_n/\partial x &= c_n W_{nn} + c_m V_{nm} e^{-\Delta}\ (\Delta = F_n - F_m). \end{aligned} \tag{3.2}$$

At the point $x = x_0$, let

$$c_m = 0,\ c_n = c_{n_0}. \tag{3.3}$$

The following formal solution of this system can be obtained for conditions (3.3):

$$c_m = e^{\int_{x_0}^{x} W_{mm}dx} \int_{x_0}^{x} c_n W_{mn} e^{-\int_{x_0}^{x} W_{mm}dx} e^{\Delta} dx,$$
$$c_n = c_{n_0} e^{\int_{x_0}^{x} W_{nn}dx} + e^{\int_{x_0}^{x} W_{nn}dx} \int_{x_0}^{x} c_m W_{nm} e^{-\int_{x_0}^{x} W_{nn}dx} e^{-\Delta} dx. \tag{3.4}$$

System (3.4) is a system of integral equations for $c_m(x)$ and $c_n(x)$, replacing the system of equations (3.2). Clearly, this is the simplest problem of exciting mode m by means of mode n. The most interesting problem here is that of exciting a Tollmien—Schlichting wave.

The transition to the system of integral equations (3.4) is quite convenient, since the eikonal is a large quantity of the order of 1/h, and to calculate in this case the integrals on the right-hand sides of (3.4) one can successfully apply the ideology of the steepest descent method (see, for example, [12]), assuming that all functions are analytic functions of the variable x.

The case called resonance excitation is of substantial interest for what follows. Let the imaginary part of the quantity α_m have the shape of Fig. 2, i.e., at the points $x > x_{ny}$ the mode m becomes unstable:

$$\mathrm{Im}\,(\alpha_m) = \frac{a}{h}(x - x_{ny}) + O\,|x - x_{ny}|^2, \tag{3.5}$$

and let the real parts of the quantities α_m and α_n be constant and equal to each other. In this case the main term in the expansion of the eikonal Δ in a series in the complex quantity $(x - x_{ny})$ is

$$\Delta = -(a/h)(x - x_{ny})^2 + O|x - x_{ny}|^3$$

(a is a real positive number). The eikonal F_m and F_n are determined here and later by the equalities

$$F_m = -i\int_{x_{ny}}^{x} \alpha_m dx, \quad F_n = -i\int_{x_{ny}}^{x} \alpha_n dx.$$

It is seen from (3.5) that the point $x = x_{ny}$ is a saddle point of the eikonal $\Delta = u + iv$, considered in the plane of the complex variable $x = x_r + ix_i$ (Fig. 3): The line u = const emerges from the point x_{ny} at an angle $\pm\pi/4$; the sectors through which the real axis passes are negative sectors. This situation also holds, obviously, when the real parts of the quantities α_m and α_n are such that

$$\mathrm{Re}\,(\alpha_m - \alpha_n) = \frac{b}{h}(x - x_{ny}) + O\,|x - x_{ny}|^2$$

(b is a real quantity) if only $|b| < a$.

The main term in h of the integral for c_m is [12], if $x_0 < x_{ny}$ and $x > x_{ny}$,

$$c_m = \sqrt{\frac{2\pi h}{a}}\, c_n(x_{ny})\, W_{mn}(x_{ny})\, e^{\int_{x_0}^{x_{ny}} W_{nm}dx} \tag{3.6}$$

Clearly, if $x < x_{ny}$, the quantity c_m vanishes in the order considered $(h^{1/2})$.

Thus, in the resonance case under consideration the wave n directly excites wave m with amplitude (3.6) at the point of stability loss of wave m.

This treatment also applies to a Tollmien—Schlichting wave if wave n has a phase velocity in the direction of the x axis which is the same as the phase velocity of the Tollmien—Schlichting wave.

4. Excitation Mechanism of a Tollmien—Schlichting Wave. The resonance excitation considered above is unstable. And what is the situation when the phase velocities of waves m and n do not coincide? To what extent are instabilities excited in this case?

Let the eikonal have the same structure as in Fig. 4. The point x^* has a saddle point of the eikonal $\Delta = u + iv$. The points A and B are the intersection points of the lines $u = \text{const}$, starting from the saddle point, with the real axis; the locations of the negative and positive sectors are shown by the corresponding signs.

It is easily verified that in the case

$$\alpha_m = \alpha_{mi}/h + (ia/h)(x - x_{ny}),\ \alpha_n = \alpha_{ni}/h.$$

for example, a similar eikonal occurs. The quantities α_{mi}, α_{ni}, a are positive constants. Then

$$\Delta = -\frac{a}{2h}(x - x_{ny})^2 + i\frac{\alpha_{mi} - \alpha_{ni}}{h}(x - x_{ny}).$$

We introduce the notations

$$\bar{x} = x/x_{ny},\quad \bar{a} = ax_{ny}^2,\quad \bar{\alpha}_{mi} = \alpha_{mi}x_{ny},$$
$$\bar{\alpha}_{ni} = \bar{\alpha}_{ni}x_{ny},\ \Delta\alpha = \bar{\alpha}_{mi} - \bar{\alpha}_{ni}.$$

The expression for Δ acquires the form

$$\Delta = -(\bar{a}/2h)(\bar{x} - 1)^2 + (i\Delta\alpha/h)(\bar{x} - 1).$$

The saddle point corresponds to $d\Delta/dx^* = 0$, $\bar{x}^* = 1 + i\Delta\alpha/\bar{a}$. The value of the eikonal Δ at the saddle point is $\Delta^* = \Delta(\bar{x}^*) = -(\Delta\alpha)^2/2h\bar{a}$. The expansion of the eikonal Δ in powers of $(x - x^*)$ acquires the form

$$\Delta = \Delta^* - (\bar{a}/2h)(x - x^*)^2.$$

The graphic representation of this example of eikonal, the eikonal portrait, is shown in Fig. 5. The lines $u = \text{const}$ passing through the saddle point, being straight, intersect the real axis at the points A and B. The hyperbolas shown correspond to the other lines $u = \text{const}$. The family of lines $v = \text{const}$ is orthogonal to the family $u = \text{const}$, and also consists of hyperbolas, reflected by the dashed lines.

We assume that all functions appearing in expression (3.4) for c_m and c_n are analytic and single-valued in the region between the real axis and the saddle point, since the integration contour, whose beginning and end are located on the real axis, can be deformed, enclosing the saddle point. The following three substantially different cases of locations of the points x_0 and x can be imagined.

1. The points x_0 and x are found inside the segment AB of Fig. 4. In this case the integration contour can be deformed into the contour c, emerging from the point x_0 and reaching the point x by the corresponding lines $v = \text{const}$, and then the values of the required integral will be the sum of integrals over small neighborhoods of the points x_0 and x. The terms of this sum will be of the order of h.

An asymptotic representation for the integral can be obtained in this case by successive integrations by parts. Indeed,

$$c_m = e^{\int_{x_0}^{x} W_{mm}dx} I,$$

$$I = \int_{x_0}^{x} c_n W_{mn} e^{-\int_{x_0}^{x} W_{mm}dx} e^{\Delta}dx = h\int_{A_1}^{B_1} \chi(h\Delta)\, e^{\Delta}d\Delta,$$

where $\chi(h\Delta) = \dfrac{c_n W_{mn}}{\dfrac{d(h\Delta)}{dx}} e^{-\int_{x_0}^{x} W_{mm}dx}$

Further

$$\int_{A_1}^{B_1} \chi(h\Delta)\, e^{\Delta}d\Delta = |e^{\Delta}\{\chi(h\Delta) - h\chi'(h\Delta) + h^2\chi'' - h^3\chi'''\}|_{A_1}^{B_1} + h^4\int_{A_1}^{B_1} \chi^{IV}e^{\Delta}d\Delta = \ldots = e^{\Delta}T(x)|_{x_0}^{x},$$

where $T(x)$ is a series in the small parameter h, starting with a term of order unity. We finally obtain for c_m

$$c_m = he^{\int_{x_0}^{x} W_{mm}dx} \{T(x)\,e^{\Delta(x)} - T(x_0)\,e^{\Delta(x_0)}\}.$$

The full expression for the m-wave is

$$\mathbf{A}_\omega^{(m)} = -h\Phi_m(x, y)\,T(x_0)\,e^{\Delta(x_0)}e^{\int_{x_0}^{x} W_{mm}dx}\,e^{F_m(x)} + h\Phi_m(x, y)\,T(x)\,e^{\int_{x_0}^{x} W_{mm}dx}\,e^{F_n(x)}. \qquad (4.1)$$

Thus, an excitation of an m-wave consists of a pure m-wave (the first term in (4.1)) and a wave (the second term) having an amplitude distribution along y as the m-wave, but with an eikonal wave n (intermediate wave).

The system of equations (3.4) for the two-mode case under consideration becomes practically uncoupled, since a correction to the quantity c_n due to wave interaction will have a relative order of smallness h^2, as the expression for the integral in the second Eq. (3.4) can be obtained by a procedure similar to that described above in obtaining expression (4.1), with the only difference that the eikonal is now replaced by $-\Delta$, and the positive sector of Fig. 4 is replaced by the negative one, and vice versa. The integration contour $x_0 - x$ does not intersect the sector boundary here either.

Therefore the quantity c_n in the function T(x) of expression (4.1) must be understood as the main term of the quantity c_{n_0}.

2. We turn now to the case in which the point x_0 is left of the point A, while the point x is inside the segment AB, i.e., the integration contour intersects the sector boundary.

The integration contour can be deformed into the sum a +b of Fig. 4. The contour a enters the saddle point along lines v =const from the side of the negative sector. The contour b emerges from the saddle point in the positive sector, and enters the point x on the real axis in the direction v =const.

The integral I over the contour a is determined by a small neighborhood ($\sim h^{1/2}$) of the saddle point, while the integral over the contour b is determined by a small neighborhood ($\sim h$) of the point x

$$I = I_a + I_b.$$

The main terms of the quantities I_a and I_b are

$$I_a = \frac{1}{2}\sqrt{\pi h}\,r_0^{-1/2}e^{i\frac{\pi-\beta_0}{2}}c_n(x^*)\,W_{mn}(x^*)\,e^{\int_{x}^{x^*} W_{mm}dx}\,e^{\Delta^*}$$

$$\left(\Delta^* = \Delta(x^*),\ \frac{1}{2}\Delta''\Big|_{x=x^*} = \frac{r_0 e^{i\beta_0}}{h}\right),$$

$$I_b = h\frac{c_n(x)\,W_{mn}(x)}{d(h\Delta)/dx}\,e^{-\int_{x_0}^{x} W_{mm}dx}\,e^{\Delta}.$$

The expression for the excited wave is

$$\mathbf{A}_\omega^{(m)} = \frac{1}{2}\sqrt{\pi h}\,\Phi_m(x, y)\,r_0^{-1/2}e^{i\frac{\pi-\beta_0}{2}}c_n(x^*)\,W_{mn}(x^*)\,e^{\Delta^*}e^{\int_{x_0}^{x^*} W_{mm}dx}\,e^{F_m(x)}$$

$$+ h\Phi_m(x, y)\frac{c_n(x)\,W_{mn}(x)}{d(h\Delta)/dx}\,e^{F_n(x)}.$$

Thus, in case 2 the excited wave consists of a pure m-wave with an amplitude independent of the point x_0, and the same intermediate wave as in case 1.

3. Let now the point x_0 be located left of point A, with point x right of point B. The integration contour can then be deformed into a +d, and, thus, the magnitude of the integral will be determined only by a small neighborhood ($\sim h^{1/2}$) of the saddle point

$$I = \sqrt{\pi h}\,r_0^{-1/2}e^{i\frac{\pi-\beta_0}{2}}c_n(x^*)\,W_{mn}(x^*)\,e^{\int_{x_0}^{x^*} W_{mm}dx}\,e^{\Delta^*}.$$

The expression for the excited wave acquires the following form of a pure m-wave:

$$A_\omega^{(m)} = \sqrt{\pi h}\,\Phi_m(x,y)\, r_0^{-1/2} e^{i\frac{\pi-\beta_0}{2}} c_n(x^*)\, W_{mn}(x^*)\, e^{\Delta^*} e^{\int_{x^*}^{x} W_{mm}dx} e^{F_m(x)}.$$

Thus, when the point x passes through the sector boundary B there appears a pure m-wave with the same amplitude as the m-wave in case 2. With further displacement of the points x_0 and x inside the negative sectors the amplitude of the perturbation wave will not change. Therefore it can be stated that the perturbation zone of the m-wave by wave n is the segment AB of the real axis, included in the positive sector of the eikonal $\Delta(x)$.

The correction to the quantity c_{n_0} in the expression for c_n due to interactions with the m-wave will be of order $h^{3/2}$ in case 3, while in case 2 it is of order h, i.e., the equations for c_m and c_n are uncoupled within the main order, and in the expression for c_m one can replace c_n by c_{n_0}.

The mechanism investigated is a mechanism of interaction of waves, having different phase velocities and identical frequencies.

It is significant that the dominant order ($\sim h^{1/2}$) of the perturbation instability (m-wave) is acquired during intersection of the integration points of neighborhoods of the points A and B, the intersection points of the lines u = const, starting from the saddle point with the real axis. At the same time the excitation has mostly the nature of resonances, similarly to the observation in Section 3: An m-wave of half-amplitude is excited by transition through point A, and the second half by transition through point B.

The effect noted can be called quasiresonance, taking into account that in this case, unlike that of true resonance in Sec. 3, the wave amplitude obtains the factor $\exp(\Delta^*)$, where $\mathrm{Re}\Delta^* < 0$.

It is important that the mechanism uncovered of instability excitation appears on a finite segment AB of variation of the variable x, located near the point of stability loss of the m-wave.

5. Propagation of a Spatial Wave Packet. Consider the single-mode regime of wave propagation in the three-dimensional boundary layer $c_1 = c \neq 0$, $c_2 = c_3 = \ldots = 0$. We investigate the development of a wave packet:

$$\Sigma A_\omega = \int_\gamma c(x, z, \gamma)\, \Phi(x, y, z, \gamma)\, e^F d\gamma. \tag{5.1}$$

The eikonal F is a large quantity ($\sim 1/h$), and is

$$F(x, z) = -i \int_{\mathscr{L}} \alpha(x, z, \gamma)\, dx + \gamma dz. \tag{5.2}$$

Since F is a large quantity ($\sim 1/h$), the behavior of the wave group under consideration is determined as a whole by the eikonal shape $F = F_r + iF_i$. We assume that all functions appearing in (5.1) are analytic functions of the parameter γ. Assuming that the contour of integration over γ can be deformed, so that it passes through the saddle point γ^* of the eikonal by negative sectors, we reach the conclusion that the value of integral (5.1) is predominantly determined by the expression [12]

$$\Sigma A_\omega = \sqrt{\pi h}\, e^{F(\gamma^*)} c(x, z, \gamma^*)\, \Phi(x, y, z, \gamma^*)\, b_0^{1/2} e^{i\frac{\pi-\beta_0}{2}} \quad \left(\frac{1}{2} F''(\gamma^*) = \frac{b_0 e^{i\beta_0}}{h}\right). \tag{5.3*}$$

The saddle point is determined by the relation

$$\partial F/\partial\gamma = 0. \tag{5.4}$$

Since the contour $\mathscr{L}$ is fixed, condition (5.4) acquires the form

*If $|b_0| = 0$, the main term of expression (5.3) will be (see [12]) of order $h^{1/4}$.

$$\int_{\mathscr{L}} \frac{\partial \alpha}{\partial \gamma}\, dx + dz = 0. \tag{5.5}$$

However, based on the fact that the point x, z is displaced along the curve $\mathscr{L}$, we conclude that

$$\partial \alpha / \partial \gamma = -dz/dx. \tag{5.6}$$

Expression (5.6) can be obtained from (5.5) on the basis of standard considerations, assuming that the integrand expression in (5.5) is nonvanishing, etc.

According to [9], expression (5.6) also determines the characteristic equation for c(x, z). We note that from our point of view the condition [9] of reality of $\partial\alpha/\partial\gamma$ is not necessary. Moreover, the fact of presence of an imaginary part of the quantity $\partial\alpha/\partial\gamma$ implies that the ordinarily solved Cauchy problem in the one-dimensional case has physical meaning, and it is necessary to solve the boundary value problem for the coefficient c(x, z).

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